Robust state reconstruction of linear neutral-type delay systems with application to lossless transmission lines: A convex optimization approach

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Abstract

This paper is concerned with the problem of robust observer design for linear systems with neutral-type time-delays. Delay-independent and delay-dependent conditions are presented to solve the observation/filtering issue under noisy output measurements. Stated as linear matrix inequalities conditions, these sufficient conditions enable the determination of the observer gains that guarantee both asymptotic convergence of the observer in case of noiseless measurements and robust filtering in case of presence of measurements errors. The proposed linear matrix inequality conditions are derived without any major approximation or assumption on the neutral type time-delay system which make the observer design straightforward and less conservative.
Keywords: Neutral-type delay systems; Observers; Optimal filtering; Linear Matrix Inequalities (LMIs).

1 INTRODUCTION

The stability and the stabilizability of neutral-type delay systems have received a revival of interest during the last decade, see for example [1], [2], [3], [4] and the references therein. Such systems appear in many practical engineering domains as distributed networks containing lossless transmission lines, chemical engineering reactor applications, ship stabilization and VLSI systems [5], [6],

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[7]. Even though considerable research efforts have been undertaken on various aspects of dynamical systems with time delays [8], [9], the observation issue of systems with neutral-type delays has received a little attention. The available results on filtering and observation of neutral time-delay systems can be broadly classified into delay dependent and delay independent techniques, see for instance [10]. Despite the fact that delay dependent observer design is considered less conservative, the delay independent techniques remain as alternative algorithms that can be used in practice.

In this paper the problem of observer design for neutral-type delay systems is addressed using the delay-independent and the delay-dependent approaches. In case of noiseless measurements, the proposed observer is merely a Luenberger observer having a classical proportional output injection term. For this case, we give sufficient linear matrix inequality condition that guarantees the existence of the observer gain. Subsequently, the problem of robustness against measurement errors is tackled. To deal with noisy measurements, the Luenberger observer is transformed into a robust observer that uses the filtered output or the integral path of the system and the observer outputs. This observer does not use the proportional output injection term as classical proportional integral observers do. For this reason, noise cannot be amplified even for high values of observer gains. Note that even the delay-independent approach does not incorporate the size of the delay in computing the observer gain, the time delay is assumed to be known to set up the observer. Illustrative example of a lossless transmission line system is used to motivate the theoretical results.

Throughout this paper, $\|\cdot\|$ stands for the usual Euclidean norm. The notation $A > 0$ (respectively $A < 0$) means that the matrix A is positive definite (respectively negative definite). We denote by $A^\top$ the matrix transpose of A. We note by I and $\mathbf{0}$ the identity matrix, and the null matrix of appropriate dimensions, respectively. "$\star$" is used to notify an element which is induced by transposition.

2 OBSERVER DESIGN

Consider the neutral-type delay system

$$\begin{aligned} &\dot{x}(t) - D\,\dot{x}(t-h) = A\,x(t) + A_d\,x(t-h) + Bu(t), \\ &y(t) = C\,x(t) + C_d\,x(t-h) + D_1\xi(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the system output. $D \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $C_d \in \mathbb{R}^{p\times n}$, and $D_1 \in \mathbb{R}^{p\times p}$ are constant matrices and h is a constant delay that appears in both the derivative and the delay state matrices. We assume that $\|D\| \leq 1$ and $\xi(t) \in \mathbb{R}^p$ is a norm-bounded uncertainty that describes usually the output measurement errors. The pair (A, C) should be either observable or detectable in the sense of Kalman. This means that the subsystem

$$\begin{aligned}\dot{\zeta}(t) &= A\,\zeta(t),\\ y(t) &= C\,\zeta(t)\end{aligned}$$

may be detectable or observable such that the Kalman's rank observability condition is verified. The formal stability of (1) is not required. However, the detectability/observability conditions imply the stability of the above system by output feedback. The initial condition of system (1) is given by

$$x(t_0+\theta) = \varphi(\theta), \;\; \forall\theta \in [-h,\; 0]\,. \tag{2}$$

The objective is to begin by designing an asymptotic observer for system (1) in the particular case where $\xi(t) \equiv 0$ and $C_d \equiv 0$. For this purpose, we set the dynamics of the observer as

$$\dot{\hat{x}}(t) - D\,\dot{\hat{x}}(t-h) = A\,\hat{x}(t) + A_d\,\hat{x}(t-h) + Bu(t) + P^{-1}Y\left(C\hat{x}(t) - y(t)\right), \tag{3}$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix and $Y \in \mathbb{R}^{n\times p}$ is a real arbitrary matrix to be determined later. Let $e(t) = \hat{x}(t) - x(t)$ be the observation error. Then, we have

$$\dot{e}(t) - D\dot{e}(t-h) = \left(A + P^{-1}YC\right)e(t) + A_d\,e(t-h). \tag{4}$$

Consider the Lyapunov-Krasovskii functional candidate

$$V(e(t)) = (e(t) - De(t-h))^\top P(e(t) - De(t-h)) + \int_{t-h}^{t} e^\top(\tau)Q\,e(\tau)\,\mathrm{d}\,\tau, \tag{5}$$

where $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ are symmetric and positive definite matrices. Then the time derivative of $V(e(t))$ along the trajectories of (4) is given by

$$\begin{aligned}&e^\top(t)\left(A^\top P + PA + YC + C^\top Y^\top + Q\right)e(t)\\ &+ e^\top(t-h)\left(-Q - D^\top PA_d - A_d^\top PD\right)e(t-h)\\ &- e^\top(t)\left(C^\top Y^\top + A^\top P\right)De(t-h)\\ &- e^\top(t-h)D^\top\left(YC + PA\right)e(t)\\ &+ e^\top(t-h)A_d^\top Pe(t) + e^\top(t)PA_d e(t-h).\end{aligned} \tag{6}$$

If we suppose that $\frac{1}{2}Q + D^\top PA_d + A_d^\top PD > 0$ then, $\dot{V}(e(t))$ can be rewritten as

$$\begin{aligned}
\dot{V} &= e^\top(t)\Big[A^\top P + PA + YC + C^\top Y^\top + Q \\
&+ PA_d\left(\frac{1}{2}Q + D^\top PA_d + A_d^\top PD\right)^{-1} A_d^\top P\Big]e(t) \\
&- \left[\left(\frac{1}{2}Q + D^\top PA_d + A_d^\top PD\right) e(t-h) - A_d^\top Pe(t)\right]^\top \\
&\times \left(\frac{1}{2}Q + D^\top PA_d + A_d^\top PD\right)^{-1} \\
&\times \left[\left(\frac{1}{2}Q + D^\top PA_d + A_d^\top PD\right) e(t-h) - A_d^\top Pe(t)\right] \\
&- e^\top(t)\left(C^\top Y^\top + A^\top P\right) De(t-h) \\
&- e^\top(t-h)D^\top (YC + PA)\, e(t) - e^\top(t-h)\frac{Q}{2}e(t-h).
\end{aligned} \tag{7}$$

This gives

$$\begin{aligned}
\dot{V} &\le e^\top(t)\Big[A^\top P + PA + YC + C^\top Y^\top + Q \\
&+ PA_d\left(\frac{1}{2}Q + D^\top PA_d + A_d^\top PD\right)^{-1} A_d^\top P\Big]e(t) \\
&- e^\top(t)\left(C^\top Y^\top + A^\top P\right) De(t-h) \\
&- e^\top(t-h)D^\top (YC + PA)\, e(t) - e^\top(t-h)\frac{Q}{2}e(t-h) \\
&= \begin{bmatrix} e(t) \\ e(t-h) \end{bmatrix}^\top \times \\
&\begin{bmatrix} \mathscr{L}_{1,1}(P, Y, Q) & -\left(C^\top Y^\top + A^\top P\right) D \\ -D^\top (YC + PA) & -\dfrac{Q}{2} \end{bmatrix} \\
&\times \begin{bmatrix} e(t) \\ e(t-h) \end{bmatrix},
\end{aligned} \tag{8}$$

where $\mathscr{L}_{1,1}(P, Y, Q) = A^\top P + PA + YC + C^\top Y^\top + Q$
$+ PA_d\left(\frac{1}{2}Q + D^\top PA_d + A_d^\top PD\right)^{-1} A_d^\top P$. Then, $\dot{V}(e(t)) < 0$ if

$$\begin{bmatrix} \mathscr{L}_{1,1}(P, Y, Q) & -\left(C^\top Y^\top + A^\top P\right) D \\ -D^\top (YC + PA) & -\dfrac{Q}{2} \end{bmatrix} < 0. \tag{9}$$

By the Schur complement lemma, the block $\mathscr{L}_{1,1}(P, Y, Q) < 0$ if

$$\begin{bmatrix} A^\top P + PA + YC + C^\top Y^\top + Q & PA_d \\ A_d^\top P & -\frac{Q}{2} - D^\top PA_d - A_d^\top PD \end{bmatrix} < 0. \tag{10}$$

Then, we can write that $\dot{V}(e(t)) < 0$ if the following linear matrix inequality holds

$$\begin{bmatrix} \mathscr{W}_{1,1}(P,Y,Q) & PA_d & -(C^\top Y^\top + A^\top P)D \\ \star & -\dfrac{Q}{2} - D^\top PA_d - A_d^\top PD & \mathbf{0} \\ \star & \star & -\dfrac{Q}{2} \end{bmatrix} < 0, \tag{11}$$

where $\mathscr{W}_{1,1}(P,Y,Q) = A^\top P + PA + YC + C^\top Y^\top + Q$.

Theorem 1 *Consider system (1) with $\xi(t) \equiv 0$ and $C_d \equiv 0$. If there exist two symmetric and positive definite matrices $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times n}$ and a matrix $Y \in \mathbb{R}^{n\times p}$ such that the linear matrix inequality (11) holds then, the states of system (3) converge asymptotically to the states of system (1) when time elapses.*

Theorem 1 gives a constructive method for designing the observer gains $K = P^{-1}Y$ via the solution of the linear matrix inequality (11) which is numerically tractable by any commercial software. Furthermore, the amount of delay does not appear in the LMI (11) which makes the observer valid for different values of the time-delay h. However, the knowledge of h remains necessary to build the dynamics of the asymptotic observer. Even though the amount of delay is not explicitly present in (11), the delay may affect considerably the performance of the observer. Remark that the condition $\frac{Q}{2} + D^\top PA_d + A_d^\top PD > 0$ that we have imposed in the previous development appears in the diagonal of the matrix of inequality (11). For this reason, it is sufficient to fulfil condition (11) to obtain the observer gains. It is important to outline that the linear matrix inequality (11) is not conservative since it is not issued from any approximation of the terms that appear in (7).

Remark 1 *When the pair (A, B) is controllable in the sense of Kalman, it is easy to extend the obtained results to get the dual conditions for the stability or the stabilizability of the neutral-delay system with full state feedback. To get this condition, it is sufficient to replace in the matrix condition of Theorem 1 the matrix A by the closed-loop matrix $A + BK$, where $K \in \mathbb{R}^{m\times n}$ and put $Y \equiv 0$.*

3 DELAY-INDEPENDENT CONDITIONS

3.1 Limitation of classical output-injection observers

Usually, the design of high-gain observers leads to noise amplification, and hence, the estimates cannot be filtered without a complete redesign of the observer gains. To clarify this fact, let us consider system (1) subject to the output uncertainty $\xi(t)$ and $C_d \equiv 0$. Then, if we use observer (3), the dynamics of the observation error is given by

$$\dot{e}(t) - D\dot{e}(t-h) = \left(A + P^{-1}YC\right) e(t) + A_d e(t-h) - P^{-1}YD_1\xi(t). \quad (12)$$

It is clear that if the stability of the observation error given by (12) requires a high-gain vector $K = P^{-1}Y$ then, the value of the uncertainty in (12) shall be amplified. For this reason, the trade off between stability and filtering remains unsolvable. Our aim is to decouple the effect of noise from the observer gains. For this purpose, we shall feed back the observer with the first integral of the system and the observer outputs. The notion of this kind of observers has been introduced in [11] for both single output linear and nonlinear systems. The reader is also referred to the references [12], [13], [14] to see other types of proportional and integral observers.

In the next subsection, we show that the output uncertainty can be translated to the system dynamics by filtering the system output and therefore, a high-gain observer design can be applied.

3.2 Robust observer design

Let us consider the augmented system

$$\begin{aligned} \dot{\eta}(t) &= \Omega\,\eta(t) + C\,x(t) + C_d\,x(t-h) + D_1\xi(t), \\ \dot{x}(t) - D\,\dot{x}(t-h) &= A\,x(t) + A_d\,x(t-h) + Bu(t), \end{aligned} \quad (13)$$

where $\eta(t)$ is considered as the new output of the neutral delay system with $\Omega \in \mathbb{R}^{p\times p}$ being Hurwitz, $x(t_0+\theta) = \varphi(\theta), \forall\theta \in [-h,\ \ 0]$, $\eta(t_0) = 0$. In the particular case where $\Omega = \mathbf{0}$ then, the new filtered output $\eta(t)$ defined as

$$\begin{aligned} \eta(t) &= \int_{t_0}^{t} \Big\{ Cx(s) + C_d\,x(s-h) + D_1\xi(s) \Big\}\ \mathrm{d}s \\ &= \int_{t_0}^{t} y(s)\ \mathrm{d}s \end{aligned} \quad (14)$$

is the integral of the original system output that is not necessarily bounded for all initial conditions. Otherwise, when Ω is Hurwitz and the system output $y(t)$ is bounded then, the new considered output $\eta(t)$ is also bounded. Let $z(t) = \begin{bmatrix} \eta(t) \\ x(t) \end{bmatrix}$ be the new state vector and define

$$\begin{aligned} \widetilde{A} &= \begin{bmatrix} \Omega & C \\ \mathbf{0} & A \end{bmatrix}, \ \widetilde{D} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix}, \\ \widetilde{D}_1 &= \begin{bmatrix} D_1 \\ \mathbf{0} \end{bmatrix}, \ \widetilde{C}^{\top} = \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix}, \\ \widetilde{B} &= \begin{bmatrix} \mathbf{0} \\ B \end{bmatrix}, \ \widetilde{A}_d = \begin{bmatrix} \mathbf{0} & C_d \\ \mathbf{0} & A_d \end{bmatrix}, \end{aligned} \quad (15)$$

as the new system matrices of dimensions $(n+p)\times(n+p)$, $(n+p)\times(n+p)$, $(n+p)\times p$, $(n+p)\times p$, $(n+p)\times m$, $(n+p)\times(n+p)$, respectively. Consider $\eta(t)$ as the new output vector of system (13). Then, we write

$$\begin{aligned} \dot{z}(t)-\widetilde{D}\dot{z}(t-h) &= \widetilde{A}z(t)+\widetilde{A}_d z(t-h)+\widetilde{B}u(t)+\widetilde{D}_1\xi(t), \\ \widetilde{y}(t) &= \widetilde{C}z(t). \end{aligned} \tag{16}$$

By taking $\eta(t)$ as the new output, we translate the uncertainty $\xi(t)$ to the state dynamics, see (16). Hence, any high-gain observer for system (16) will act as a filter because noise is viewed now as a system uncertainty. Remark that when the output contains state-delayed terms, the delay states will be also translated to the system dynamics, keeping the new output $\eta(t)$ free from any state delay.

Consequently, the dynamics of the observer is readily constructed as

$$\dot{\hat{z}}(t)-\widetilde{D}\dot{\hat{z}}(t-h)=\widetilde{A}\hat{z}(t)+\widetilde{A}_d\hat{z}(t-h)+\widetilde{B}u(t)+\widetilde{P}^{-1}\widetilde{Y}\left(\widetilde{C}\hat{z}(t)-\widetilde{y}(t)\right), \tag{17}$$

where $\widetilde{P}^{-1}\widetilde{Y}$ is the observer gain of appropriate dimensions . Hence, the dynamics of the observation error $e(t)=\hat{z}(t)-z(t)$ becomes

$$\dot{e}(t)-\widetilde{D}\dot{e}(t-h)=\left(\widetilde{A}+\widetilde{P}^{-1}\widetilde{Y}\widetilde{C}\right)e(t)+\widetilde{A}_d e(t-h)-\widetilde{D}_1\xi(t). \tag{18}$$

Even though the new observer dynamics (17) is in form (4), the injection term of (17) is an integral path of the difference of the observer and the system outputs. It remains now to deal with the calculation of the observer gains so as to ensure the asymptotic stability of the observation error when $\xi(t)\equiv 0$ and satisfy the following performance index for all initial conditions $e(s),\ -h\le s\le 0$ and $\forall t\ge 0$

$$\int_0^t\left\{e^\top(s)\widetilde{C}^\top\widetilde{C}e(s)-\gamma^2\xi^\top(s)\xi(s)\right\}\,\mathrm{d}\,s\le\mathscr{V}(0); \tag{19}$$

where $\mathscr{V}(0)=(e(0)-\widetilde{D}e(-h))^\top\widetilde{P}(e(0)-\widetilde{D}e(-h))+\int_{-h}^0 e^\top(\tau)\widetilde{Q}e(\tau)\mathrm{d}\,\tau$, and $\widetilde{P}$, $\widetilde{Q}$ are symmetric and positive definite matrices of appropriate dimensions. It is obvious that if the initial conditions $e(t)=0$ for $-h\le t\le 0$, then the performance index (19)is equivalent to $\|\widetilde{C}e(t)\|\le\gamma\|\xi(t)\|$. Setting the performance index in form (19) is realistic since the initial condition of the system is generally unknown in such observation problems. We summarize the result of this section in the following statement.

Theorem 2 *The observer error dynamics (18) is asymptotically stable for $\xi(t)\equiv 0$ and verifies condition (19) for $\xi(t)\not\equiv 0$ if there exist two positive and definite* *matrices* $\widetilde{P} \in \mathbb{R}^{(n+p)\times(n+p)}$, $\widetilde{Q} \in \mathbb{R}^{(n+p)\times(n+p)}$*, a matrix* $\widetilde{Y} \in \mathbb{R}^{(n+p)\times p}$*, and a positive constant* γ^2 *such that the following LMI holds*

$$\begin{bmatrix} \mathscr{M}_{1,1} & \widetilde{P}\widetilde{A}_d & -(\widetilde{C}^\top\widetilde{Y}^\top + \widetilde{A}^\top\widetilde{P})\widetilde{D} & -\widetilde{P}\widetilde{D}_1 \\ \star & -\frac{\widetilde{Q}}{2} - \widetilde{D}^\top\widetilde{P}\widetilde{A}_d - \widetilde{A}_d^\top\widetilde{P}\widetilde{D} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\frac{\widetilde{Q}}{2} & \widetilde{D}^\top\widetilde{P}\widetilde{D}_1 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} < 0, \tag{20}$$

where $\mathscr{M}_{1,1} = \widetilde{A}^\top\widetilde{P} + \widetilde{P}\widetilde{A} + \widetilde{Y}\widetilde{C} + \widetilde{C}^\top\widetilde{Y}^\top + \widetilde{C}^\top\widetilde{C} + \widetilde{Q}$*.*

Proof. Let $\mathscr{V}(e(t)) = (e(t)-\widetilde{D}e(t-h))^\top\widetilde{P}(e(t)-\widetilde{D}e(t-h))+\int_{t-h}^{t} e^\top(\tau)\widetilde{Q}e(\tau)\,\mathrm{d}\,\tau$. Then, we have

$$\begin{aligned} &\int_0^t \left\{e^\top(s)\widetilde{C}^\top\widetilde{C}e(s) - \gamma^2\xi^\top(s)\xi(s)\right\}\,\mathrm{d}\,s - \mathscr{V}(0) \\ &\leq \int_0^t \left\{e^\top(s)\widetilde{C}^\top\widetilde{C}e(s) - \gamma^2\xi^\top(s)\xi(s)\right\}\,\mathrm{d}\,s + \mathscr{V}(e(t)) - \mathscr{V}(0) \\ &= \int_0^t \left\{e^\top(s)\widetilde{C}^\top\widetilde{C}e(s) - \gamma^2\xi^\top(s)\xi(s) + \dot{\mathscr{V}}(e(s))\right\}\,\mathrm{d}\,s. \end{aligned} \tag{21}$$

Under the assumption that $\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top \widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top \widetilde{P}\widetilde{D} > 0$, we have

$$
\begin{aligned}
& e^\top(t)\widetilde{C}^\top \widetilde{C} e(t) - \gamma^2 \xi^\top(t)\xi(t) + \dot{\mathscr{V}}(e(t)) \\
& = e^\top(t)\widetilde{C}^\top \widetilde{C} e(t) - \gamma^2 \xi^\top(t)\xi(t) \\
& + e^\top(t)\Big[A^\top \widetilde{P} + \widetilde{P}A + \widetilde{Y}\widetilde{C} + \widetilde{C}^\top \widetilde{Y}^\top + \widetilde{Q} + \widetilde{P}\widetilde{A}_d \\
& \times \Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top \widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top \widetilde{P}\widetilde{D}\Big)^{-1} \widetilde{A}_d^\top \widetilde{P}\Big] e(t) \\
& - \Big[\Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top \widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top \widetilde{P}\widetilde{D}\Big) e(t-h) - \widetilde{A}_d^\top \widetilde{P} e(t)\Big]^\top \\
& \times \Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top \widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top \widetilde{P}\widetilde{D}\Big)^{-1} \times \\
& \Big[\Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top \widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top \widetilde{P}\widetilde{D}\Big) e(t-h) - \widetilde{A}_d^\top \widetilde{P} e(t)\Big] \\
& - e^\top(t)\Big(\widetilde{C}^\top \widetilde{Y}^\top + \widetilde{A}^\top \widetilde{P}\Big)\widetilde{D} e(t-h) \\
& - e^\top(t-h)\widetilde{D}^\top \Big(\widetilde{Y}\widetilde{C} + \widetilde{P}\widetilde{A}\Big) e(t) \\
& - e^\top(t-h)\frac{\widetilde{Q}}{2} e(t-h) \\
& - \xi^\top(t)\widetilde{D}_1^\top \widetilde{P} e(t) + \xi^\top(t)\widetilde{D}_1^\top \widetilde{P}\widetilde{D} e(t-h) \\
& - e^\top(t)\widetilde{P}\widetilde{D}_1 \xi(t) + e^\top(t-h)\widetilde{D}^\top \widetilde{P}\widetilde{D}_1 \xi(t) \\
& \leq \zeta^\top(t) \begin{bmatrix} \mathscr{L}_{1,1}(\widetilde{P}, \widetilde{Y}, \widetilde{Q}) + \widetilde{C}^\top \widetilde{C} & -(\widetilde{C}^\top \widetilde{Y}^\top + \widetilde{A}^\top \widetilde{P})\widetilde{D} & -\widetilde{P}\widetilde{D}_1 \\ -\widetilde{D}^\top(\widetilde{Y}\widetilde{C} + \widetilde{P}\widetilde{A}) & -\frac{\widetilde{Q}}{2} & \widetilde{D}^\top \widetilde{P}\widetilde{D}_1 \\ -\widetilde{D}_1^\top \widetilde{P} & \widetilde{D}_1^\top \widetilde{P}\widetilde{D} & -\gamma^2 I \end{bmatrix} \zeta(t),
\end{aligned} \tag{22}
$$

where $\zeta(t) = \begin{bmatrix} e^\top(t) & e^\top(t-h) & \xi^\top(t) \end{bmatrix}^\top$, and $\mathscr{L}_{1,1}(\widetilde{P}, \widetilde{Y}, \widetilde{Q}) = A^\top \widetilde{P} + \widetilde{P}A + \widetilde{Y}\widetilde{C} + \widetilde{C}^\top \widetilde{Y}^\top + \widetilde{Q} + \widetilde{P}\widetilde{A}_d \Big(\frac{1}{2}\widetilde{Q} + \widetilde{D}^\top \widetilde{P}\widetilde{A}_d + \widetilde{A}_d^\top \widetilde{P}\widetilde{D}\Big)^{-1} \widetilde{A}_d^\top \widetilde{P}$. Then the optimality condition (19) is satisfied if

$$
\begin{bmatrix} \mathscr{L}_{1,1}(\widetilde{P}, \widetilde{Y}, \widetilde{Q}) + \widetilde{C}^\top \widetilde{C} & -(\widetilde{C}^\top \widetilde{Y}^\top + \widetilde{A}^\top \widetilde{P})\widetilde{D} & -\widetilde{P}\widetilde{D}_1 \\ -\widetilde{D}^\top(\widetilde{Y}\widetilde{C} + \widetilde{P}\widetilde{A}) & -\frac{\widetilde{Q}}{2} & \widetilde{D}^\top \widetilde{P}\widetilde{D}_1 \\ -\widetilde{D}_1^\top \widetilde{P} & \widetilde{D}_1^\top \widetilde{P}\widetilde{D} & -\gamma^2 I \end{bmatrix} < 0. \tag{23}
$$

By applying the Schur complement result, condition (23) is equivalent to (20). It is always interesting to find the smallest value of γ that verifies inequality (20). In this case, the linear optimization problem (20) must be modified to $\min_{\widetilde{P}, \widetilde{Y}, \widetilde{Q}} \ \gamma^2 \ \text{ s.t. } (20)$.

In order to solve this optimization problem, the parameter γ^2 can be replaced by another parameter $p > 0$, with $p < p_{\max}$ where $p_{\max}$ is given. We can also leave p as an LMI variable to be obtained by the software.

4 DELAY-DEPENDENT APPROACH

In this Section, delay-dependent conditions are discussed. The main objective is to study the convergence of the observer/filter and its ability to reduce the amount of uncertainties/noise that may contain the estimates. In order to achieve this objective, a filtered output is used to feed back the observer as it is introduced in the previous section. Consider again system (16) where all the state matrices are defined as in (15). Similar to the delay-independent case, we propose an observer of the following form:

$$\dot{\hat{z}}(t) - \widetilde{D}\,\dot{\hat{z}}(t-h) = \widetilde{A}\,\hat{z}(t) + \widetilde{A}_d\,\hat{z}(t-h) + \widetilde{B}u(t) + P^{-1}Y\left(\widetilde{C}\hat{z}(t) - \tilde{y}(t)\right) \tag{24}$$

where $P = P^\top > 0$ and Y are real matrices of appropriate dimensions to be defined later. The computation of the observer gains $P^{-1}Y$ in order to assure an attenuation level of uncertainty is discussed in the following statement.

Theorem 3 *Consider systems (16) and (24) and let $e(t) = \hat{z}(t) - z(t)$ be the observation error between the two systems. If for given γ^2 there exist five symmetric and positive definite matrices P, Q, Q_1, Q_2, Z of appropriate dimensions and a real matrix Y such that the following matrix inequality holds*

$$\begin{bmatrix} \mathscr{L}_{11} & \mathscr{L}_{12} & \mathscr{L}_{13} & \mathscr{L}_{14} & \mathscr{L}_{15} & \mathscr{L}_{16} & \mathscr{L}_{17} \\ \star & \mathscr{L}_{22} & \mathscr{L}_{23} & \mathscr{L}_{24} & \mathscr{L}_{25} & \mathscr{L}_{26} & \mathscr{L}_{27} \\ \star & \star & \mathscr{L}_{33} & \mathscr{L}_{34} & \mathscr{L}_{35} & \mathscr{L}_{36} & \mathscr{L}_{37} \\ \star & \star & \star & \mathscr{L}_{44} & \mathscr{L}_{45} & \mathscr{L}_{46} & \mathscr{L}_{47} \\ \star & \star & \star & \star & \mathscr{L}_{55} & \mathscr{L}_{56} & \mathscr{L}_{57} \\ \star & \star & \star & \star & \star & \mathscr{L}_{66} & \mathscr{L}_{67} \\ \star & \star & \star & \star & \star & \star & \mathscr{L}_{77} \end{bmatrix} < 0 \tag{25}$$

then, the observation error is globally asymptotically stable for $\xi(t) \equiv 0$ and for the null initial conditions, i,e, $x(\tau) = \hat{x}(\tau)$ for $-h \leq \tau \leq 0$, the following integral inequality

$$\int_0^t \left\{ e^\top(v)\widetilde{C}^\top\widetilde{C}e(v) - \gamma^2\xi^\top(v)\xi(v) \right\} \mathrm{d}v < 0 \tag{26}$$

is verified for all $t \geq 0$ *where*

$$
\begin{aligned}
\mathscr{L}_{11} &\triangleq P\,\left(\widetilde{A}_c + \widetilde{A}_d\right) + \left(\widetilde{A}_c^\top + \widetilde{A}_d^\top\right) P + Q + h\,\widetilde{A}_c^\top Q_1\,\widetilde{A}_c + \widetilde{C}^\top \widetilde{C},\\
\mathscr{L}_{12} &\triangleq -\left(\widetilde{A}_c^\top + \widetilde{A}_d^\top\right) P\,\widetilde{D} + h\widetilde{A}_c^\top Q_1\,\widetilde{A}_d,\\
\mathscr{L}_{13} &\triangleq h\widetilde{A}_c^\top Q_1\,\widetilde{D},\\
\mathscr{L}_{14} &\triangleq -hP\widetilde{A}_d,\\
\mathscr{L}_{15} &\triangleq \mathbf{0},\\
\mathscr{L}_{16} &\triangleq -P\,\widetilde{D}_1 - h\widetilde{A}_c^\top Q_1\,\widetilde{D}_1,\\
\mathscr{L}_{17} &\triangleq -h\widetilde{A}_c^\top Z,\\
\mathscr{L}_{22} &\triangleq -Q + h\widetilde{A}_d^\top Q_1\,\widetilde{A}_d,\\
\mathscr{L}_{23} &\triangleq h\widetilde{A}_d^\top Q_1\,\widetilde{D},\\
\mathscr{L}_{24} &\triangleq h\widetilde{D}^\top P\widetilde{A}_d,\\
\mathscr{L}_{25} &\triangleq \mathbf{0},\\
\mathscr{L}_{26} &\triangleq \widetilde{D}^\top P\,\widetilde{D}_1 - h\widetilde{A}_d^\top Q_1\,\widetilde{D}_1\\
\mathscr{L}_{27} &\triangleq h\widetilde{A}_d^\top Z,\\
\mathscr{L}_{33} &\triangleq -hQ_1 + hQ_2 + h\widetilde{D}^\top Q_1\,\widetilde{D},\\
\mathscr{L}_{34} &\triangleq \mathbf{0},\\
\mathscr{L}_{35} &\triangleq \mathbf{0},\\
\mathscr{L}_{36} &\triangleq -h\widetilde{D}^\top Q_1\,\widetilde{D}_1,\\
\mathscr{L}_{37} &\triangleq \mathbf{0},\\
\mathscr{L}_{44} &\triangleq -hZ,\\
\mathscr{L}_{45} &\triangleq hZ\widetilde{D},\\
\mathscr{L}_{46} &\triangleq \mathbf{0},\\
\mathscr{L}_{47} &\triangleq \mathbf{0},\\
\mathscr{L}_{55} &\triangleq -h\widetilde{D}^\top Z\,\widetilde{D} - hQ_2,\\
\mathscr{L}_{56} &\triangleq \mathbf{0},\\
\mathscr{L}_{57} &\triangleq \mathbf{0},\\
\mathscr{L}_{66} &\triangleq h\widetilde{D}_1^\top Q_1\,\widetilde{D}_1 - \gamma^2\,I,\\
\mathscr{L}_{67} &\triangleq -hD_1^\top Z,\\
\mathscr{L}_{77} &\triangleq -hZ,
\end{aligned}
$$

and $\widetilde{A}_c = \widetilde{A} + P^{-1}Y\widetilde{C}$*.*

Proof. See the Appendix Section.

In contrast to delay-independent conditions, the delay-dependent conditions given in the statement of Theorem 3 seem to be less conservative due to the fact that small delay h enhances the negativity of the matrix condition. This means that the knowledge of the delay size is necessary to build the observer. This leads, however, to the conclusion that the proposed observer is not robust with respect to eventual delay uncertainties.

Remark 2 *The matrix inequality condition of Theorem 3 is not convex, but it can be solved in iterative manner. However, the problem can be made convex by putting $Q_1 = P$. This may introduce certain conservatism in the LMI condition, but on the other hand, the numerical computation becomes easier with use of convex-optimization software, see the references [15], [16] for more details on LMI design and computation.*

Remark 3 *From the previous developments, we see that the delay-dependent conditions are derived without bounding the cross terms that appear from the derivative of the proposed Lyapunov functionals. This consequently helps in revealing the conservatism of the conditions under which the filter is convergent with uncertainty attenuation.*

Remark 4 *By considering the new output $\eta(t)$ as the solution of*

$$\dot{\eta}(t) = \Omega\,\eta(t) + y(t), \tag{27}$$

the delayed states that may contain the original system output are also transferred to the equations of the system dynamics leaving the observer output injection free from both the output uncertainty and the delay h. In the case where the output $y(t)$ contains state variables with different time-delays, the resulting augmented system can be seen as neutral-type system with multiple delays.

5 ILLUSTRATIVE EXAMPLE

The dynamics of a lossless transmission line is modelled by the following neutral-type delay system [6]

$$
\begin{aligned}
\dot{x}(t) - \begin{bmatrix} 0 & \alpha_4 & 0 & 0 \\ \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x}(t-h) &= \begin{bmatrix} -\alpha_0 & 0 & \alpha_0 & 0 \\ 0 & -\alpha_1 & 0 & -\alpha_1 \\ -\alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_3 & 0 & 0 \end{bmatrix} x(t) \\
&+ \begin{bmatrix} 0 & \alpha_0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2\,\alpha_4 & 0 & 0 \\ -\alpha_3\,\alpha_5 & 0 & 0 & 0 \end{bmatrix} x(t-h) + \begin{bmatrix} 0 & \beta_0 \\ 0 & 0 \\ \alpha_2\beta_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\
y(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \xi(t),
\end{aligned}
\tag{28}
$$

where the system parameters are defined as $\alpha_0 = \sqrt{c}/(c_0 R_0 \sqrt{c} + c_0\sqrt{L})$, $\alpha_1 = \sqrt{c}/(c_1 R_1 \sqrt{c} + c_1 \sqrt{L})$, $\alpha_2 = (R_0\sqrt{c} + \sqrt{L})/(L_0\sqrt{c})$, $\alpha_3 = (R_1\sqrt{c_1} + \sqrt{L})/(L_1\sqrt{c})$, $\alpha_4 = (R_0\sqrt{c} - \sqrt{L})/(R_0\sqrt{c} + \sqrt{L})$, $\alpha_5 = (R_1\sqrt{c} - \sqrt{L})/(R_1\sqrt{c} + \sqrt{L})$, $u_2(t) = \dot{u}_1(t)$, $h = \sqrt{c\,L}$. The numerical values of the system parameters are: $L_1 = 0.2$ $[H]$, $L = 1$ $[m]$, $h = 0.1414$ $[S]$, $c = 0.02$ $[S^2/m]$, $L_0 = 0.1$ $[H]$, $\beta_0 = 0.01$, $d_1 = 0.3$, $d_2 = 0.1$, $c_1 = 0.1$ $[F]$, $R_0 = 5$ $[Ohms]$, $R_1 = 10$ $[Ohms]$. To implement the robust time-delay observer (17) we shall delay the observer states by a constant delay h and consider the terms $\widetilde{A}_d\hat{z}(t-h)$ and $\widetilde{D}\dot{\hat{z}}(t-h)$ as feedback inputs to observer (17). This technique permits us to implement the observer dynamics as it appears without augmenting the order of the states. In this simulation an integral action is taken by putting $\Omega = \mathbf{0}$. In Fig. 1 the noisy output of system (28) are reported when a periodic control input $u_1(t) = 5\,\sin(10\,t)$ [V] is applied to system (28). The initial values of system (28) are $(x_i)_{1\leq i\leq 4}(t) = -1$ for $t < h$. In Figs. 2 and 3, the behavior of the estimated states along with the real states of system (28) are represented. From these simulations, we see clearly that the observer states are quite insensitive to a band-limited white noise that comes corrupting the measurements. The maximum amplitude of noise is set to 10. The simulations are done after solving the filtering problem (20) with respect to P, Q, Y and γ. The solution is

$$P = \begin{bmatrix} 4.2567 & 0 & 0.4138 & 0 & -0.2236 & 0 \\ 0 & 7.2728 & 0 & -0.8400 & 0 & -0.3903 \\ 0.4138 & 0 & 2.5476 & 0 & 0.0363 & 0 \\ 0 & -0.8400 & 0 & 2.7913 & 0 & -0.0538 \\ -0.2236 & 0 & 0.0363 & 0 & 0.0195 & 0 \\ 0 & -0.3903 & 0 & -0.0538 & 0 & 0.0539 \end{bmatrix},$$

$$Q = \begin{bmatrix} 2.5839 & 0 & -0.1405 & 0 & -0.0070 & 0 \\ 0 & 2.5769 & 0 & 0.1190 & 0 & -0.0049 \\ -0.1405 & 0 & 6.0484 & 0 & 0.1108 & 0 \\ 0 & 0.1190 & 0 & 6.2529 & 0 & -0.0932 \\ -0.0070 & 0 & 0.1108 & 0 & 0.2461 & 0 \\ 0 & -0.0049 & 0 & -0.0932 & 0 & 0.3951 \end{bmatrix},$$

$$Y = \begin{bmatrix} -7.4319 & 0 \\ 0.0001 & -6.2441 \\ -18.2621 & 0.0002 \\ 0 & 18.4801 \\ -3.8303 & 0 \\ 0 & -6.8959 \end{bmatrix}, \ \gamma = 0.8803.$$

6 CONCLUSION

The problem of robust observer design for a class of systems with neutral-type time delays is addressed. Delay-dependent and delay-independent LMI conditions are derived and their numerical tractability are discussed. Accordingly, extension of this work to neutral systems with multiple time-delays is also possible. We conjecture that dual results can be obtained in case of stabilization by static feedback and more optimality conditions can be imposed. This point will be the subject of future investigation.

Appendix

Proof of Theorem 3

From (16) and (24), define $e(t) = \hat{z}(t) - z(t)$. Let us consider the following

Lyapunov-Krasovskii functional

$$\begin{aligned}
V_1 &\triangleq \Big[e(t) - \widetilde{D}\, e(t-h)\Big]^\top P \Big[e(t) - \widetilde{D}\, e(t-h)\Big], \\
V_2 &\triangleq \int_{t-h}^{t} e^\top(s)\, Q\, e(s)\, \mathrm{d}s, \\
V_3 &\triangleq \int_{-h}^{0}\int_{t+\theta}^{t} \Big[\dot{e}(v) - \widetilde{D}\, \dot{e}(v-h)\Big]^\top Z \Big[\dot{e}(v) - \widetilde{D}\, \dot{e}(v-h)\Big] \mathrm{d}\theta \mathrm{d}v, \\
V_4 &\triangleq h \int_{t-h}^{t} \dot{e}^\top(s) Q_1 \dot{e}(s)\, \mathrm{d}s, \\
V_5 &= \int_{-h}^{0}\int_{t+\theta}^{t} \dot{e}^\top(v-h)\, Q_2\, \dot{e}(v-h)\, \mathrm{d}\theta \mathrm{d}v.
\end{aligned} \tag{29}$$

As we have proved in the previous section, the integral inequality (26) is satisfied if the following holds

$$e^\top(t)\widetilde{C}^\top \widetilde{C} e(t) - \gamma^2 \xi^\top(t)\xi(t) + \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 + \dot{V}_5 < 0. \tag{30}$$

Let

$$\zeta(t,v) \triangleq \begin{bmatrix} e(t) \\ e(t-h) \\ \dot{e}(t-h) \\ \dot{e}(v) \\ \dot{e}(v-h) \\ \xi(t) \end{bmatrix} \tag{31}$$

then, we have

$$\dot{V}_1 = 2\Big[e^\top(t) - e^\top(t-h)\widetilde{D}^\top\Big] P \Big[\dot{e}(t) - \widetilde{D}\, \dot{e}(t-h)\Big]. \tag{32}$$

Using the fact that

$$e(t) - e(t-h) = \int_{t-h}^{t} \dot{e}(s)\, \mathrm{d}s,$$

then, we can write that

$$\begin{aligned}\dot{V}_1 &= 2\Big[e^\top(t) - e^\top(t-h)\widetilde{D}^\top\Big]P\Big[(\widetilde{A}_c + \widetilde{A}_d)e(t) - \widetilde{A}_d\int_{t-h}^{t}\dot{e}(s)\,\mathrm{d}s - \widetilde{D}_1\xi(t)\Big].\\
&= \frac{1}{h}\int_{t-h}^{t} 2e^\top(t)P(\widetilde{A}_c+\widetilde{A}_d)e(t)\,\mathrm{d}v\\
&+ \frac{1}{h}\int_{t-h}^{t} 2e^\top(t)(-h\,P\,\widetilde{A}_d)\dot{e}(v)\,\mathrm{d}v + \frac{1}{h}\int_{t-h}^{t} 2e^\top(t)(-P\,\widetilde{D}_1)\xi(t)\,\mathrm{d}v\\
&+ \frac{1}{h}\int_{t-h}^{t} 2e^\top(t-h)\Big(-\widetilde{D}^\top P(\widetilde{A}_c+\widetilde{A}_d)\Big)e(t)\,\mathrm{d}v\\
&+ \frac{1}{h}\int_{t-h}^{t} 2e^\top(t-h)(h\widetilde{D}^\top P\widetilde{A}_d)\dot{e}(v)\,\mathrm{d}v + \frac{1}{h}\int_{t-h}^{t} 2e^\top(t-h)(\widetilde{D}^\top P\widetilde{D}_1)\xi(t)\,\mathrm{d}v\\
&= \frac{1}{h}\int_{t-h}^{t}\zeta^\top(t,v)\Omega_1\zeta(t,v)\,\mathrm{d}v
\end{aligned} \tag{33}$$

where

$$\Omega_1 = \begin{bmatrix} (\widetilde{A}_c+\widetilde{A}_d)^\top P + P(\widetilde{A}_c+\widetilde{A}_d) & -(\widetilde{A}_c+\widetilde{A}_d)^\top P\widetilde{D} & \mathbf{0} & -hP\widetilde{A}_d & 0 & -P\widetilde{D}_1\\ \star & \mathbf{0} & \mathbf{0} & h\widetilde{D}^\top P\widetilde{A}_d & \mathbf{0} & \widetilde{D}^\top P\widetilde{D}_1\\ \star & \star & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \star & \star & \star & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \star & \star & \star & \star & \mathbf{0} & \mathbf{0}\\ \star & \star & \star & \star & \star & \mathbf{0}\end{bmatrix}. \tag{34}$$

On the other hand,

$$\begin{aligned}\dot{V}_2 &= e^\top(t)Qe(t) - e^\top(t-h)Qe(t-h)\\ &= \frac{1}{h}\int_{t-h}^{t}\zeta^\top(t,v)\,\Omega_2\,\zeta(t,v)\,\mathrm{d}v\end{aligned} \tag{35}$$

where

$$\Omega_2 = \mathrm{diag}\Big(Q, -Q, \mathbf{0},\, \mathbf{0},\, \mathbf{0},\, \mathbf{0}\Big). \tag{36}$$

We have

$$
\begin{aligned}
\dot{V}_3 &= h\Big[\dot{e}(t) - \widetilde{D}\dot{e}(t-h)\Big]^\top Z\Big[\dot{e}(t) - \widetilde{D}\dot{e}(t-h)\Big] \\
&\quad - \frac{1}{h}\int_{t-h}^{t} \Big[\dot{e}(v) - \widetilde{D}\dot{e}(v-h)\Big]^\top (hZ)\Big[\dot{e}(v) - \widetilde{D}\dot{e}(v-h)\Big]\,\mathrm{d}v \\
&= \frac{1}{h}\int_{t-h}^{t} \Big[\dot{e}(t) - \widetilde{D}\dot{e}(t-h)\Big]^\top (hZ)\Big[\dot{e}(t) - \widetilde{D}\dot{e}(t-h)\Big]\,\mathrm{d}v \\
&\quad + \frac{1}{h}\int_{t-h}^{t} \Big[\dot{e}(v) - \widetilde{D}\dot{e}(v-h)\Big]^\top (-hZ)\Big[\dot{e}(v) - \widetilde{D}\dot{e}(v-h)\Big]\,\mathrm{d}v.
\end{aligned}
\tag{37}
$$

This gives

$$
\begin{aligned}
\dot{V}_3 = \frac{1}{h}\int_{t-h}^{t} \zeta^\top(t,v) \begin{bmatrix} \widetilde{A}_c^\top \\ \widetilde{A}_d^\top \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\widetilde{D}_1^\top \end{bmatrix} (h\,Z) \begin{bmatrix} \widetilde{A}_c & \widetilde{A}_d & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\widetilde{D}_1 \end{bmatrix} \zeta(t,v)\,\mathrm{d}v \\
+ \frac{1}{h}\int_{t-h}^{t} \zeta^\top(t,v)\,\Omega_{32}\,\zeta(t,v)\,\mathrm{d}v
\end{aligned}
\tag{38}
$$

where

$$
\Omega_{32} \triangleq \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ I \\ -\widetilde{D}^\top \\ \mathbf{0} \end{bmatrix} (-hZ) \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & I & -\widetilde{D} & \mathbf{0} \end{bmatrix}. \tag{39}
$$

By differentiating V_4, we get

$$
\begin{aligned}
\dot{V}_4 &= h\,\dot{e}^\top(t)Q_1\dot{e}(t) - h\,\dot{e}^\top(t-h)Q_1\dot{e}(t-h) \\
&= \frac{1}{h}\int_{t-h}^{t} \dot{e}^\top(t)\;(h\,Q_1)\;\dot{e}(t)\;\mathrm{d}v \\
&\quad + \frac{1}{h}\int_{t-h}^{t} \dot{e}^\top(t-h)\;(-h\,Q_1)\;\dot{e}(t-h)\;\mathrm{d}v.
\end{aligned}
\tag{40}
$$

Using the equations of the system dynamics, we find

$$\begin{aligned}\dot{V}_4 &= \frac{1}{h}\int_{t-h}^{t} \Big(\widetilde{A}_c\, e(t) + \widetilde{A}_d\, e(t-h) + \widetilde{D}\dot{e}(t-h) - \widetilde{D}_1\xi(t)\Big)^{\top} \\ &\times (h\, Q_1)\Big(\widetilde{A}_c\, e(t) + \widetilde{A}_d\, e(t-h) + \widetilde{D}\dot{e}(t-h) - \widetilde{D}_1\xi(t)\Big)\, \mathrm{d}v \\ &+ \frac{1}{h}\int_{t-h}^{t} \dot{e}^{\top}(t-h)\ (-h\, Q_1)\ \dot{e}(t-h)\, \mathrm{d}v.\end{aligned} \tag{41}$$

Finally, we write

$$\dot{V}_4 = \frac{1}{h}\int_{t-h}^{t} \zeta^{\top}(t,v)\Omega_4\zeta(t,v)\, \mathrm{d}v, \tag{42}$$

where

$$\begin{aligned}\Omega_4 &\triangleq \begin{bmatrix} \widetilde{A}_c^{\top} \\ \widetilde{A}_d^{\top} \\ \widetilde{D}^{\top} \\ \mathbf{0} \\ \mathbf{0} \\ -\widetilde{D}_1^{\top} \end{bmatrix} (h\, Q_1) \begin{bmatrix} \widetilde{A}_c & \widetilde{A}_d & \widetilde{D} & \mathbf{0} & \mathbf{0} & -\widetilde{D}_1 \end{bmatrix} \\ &+ \mathrm{diag}\Big(\mathbf{0}, \mathbf{0}, -h\, Q_1, \mathbf{0}, \mathbf{0}, \mathbf{0}\Big).\end{aligned} \tag{43}$$

We have

$$\begin{aligned}\dot{V}_5 &= h\, \dot{e}^{\top}(t-h)\, Q_2\, \dot{e}(t-h) - \int_{t-h}^{t} \dot{e}^{\top}(v-h)\, Q_2\, \dot{e}(v-h) \\ &= \frac{1}{h}\int_{t-h}^{t} \zeta^{\top}(t,v)\Omega_5\zeta(t,v)\, \mathrm{d}v,\end{aligned} \tag{44}$$

where

$$\Omega_5 = \mathrm{diag}\Big(\mathbf{0}, \mathbf{0}, h\, Q_2, \mathbf{0}, \mathbf{0}, -h\, Q_2\Big). \tag{45}$$

As a result

$$\begin{aligned}\dot{V} &= \frac{1}{h}\int_{t-h}^{t} \zeta^{\top}(t,v)\big(\Omega_1 + \Omega_2 + \Omega_{32} + \Omega_4 + \Omega_5\big)\zeta(t,v)\, \mathrm{d}v, \\ &+ \frac{1}{h}\int_{t-h}^{t} \zeta^{\top}(t,v) \begin{bmatrix} \widetilde{A}_c^{\top} \\ \widetilde{A}_d^{\top} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\widetilde{D}_1^{\top} \end{bmatrix} (h\, Z) \begin{bmatrix} \widetilde{A}_c & \widetilde{A}_d & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\widetilde{D}_1 \end{bmatrix} \zeta(t,v)\, \mathrm{d}v.\end{aligned} \tag{46}$$

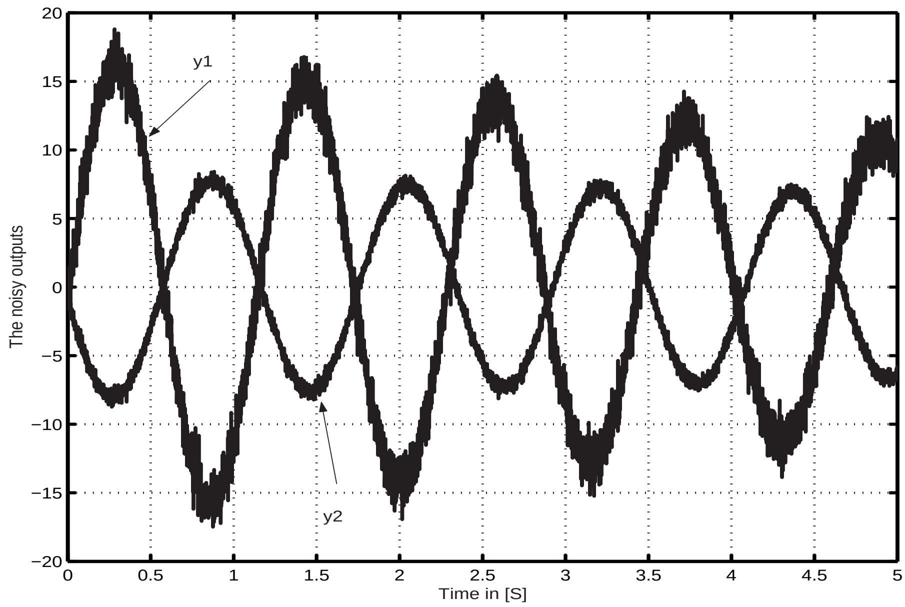

Figure 1: The Noisy outputs

This immediately implies that inequality (30) is satisfied if the following matrix inequality holds

$$\begin{bmatrix} \widetilde{C}^\top \widetilde{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -\gamma^2 I & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & \mathbf{0} \end{bmatrix} + \left[\begin{array}{c|c} \Omega_1 + \Omega_2 + \Omega_{32} + \Omega_4 + \Omega_5 & \begin{array}{c} h\widetilde{A}_c^\top Z \\ h\,\widetilde{A}_d^\top Z \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -h\,\widetilde{D}_1^\top Z \end{array} \\ \hline \star & -h\,Z \end{array}\right] < 0. \tag{47}$$

By rearranging the last inequality, we find inequality (25). This ends the proof.

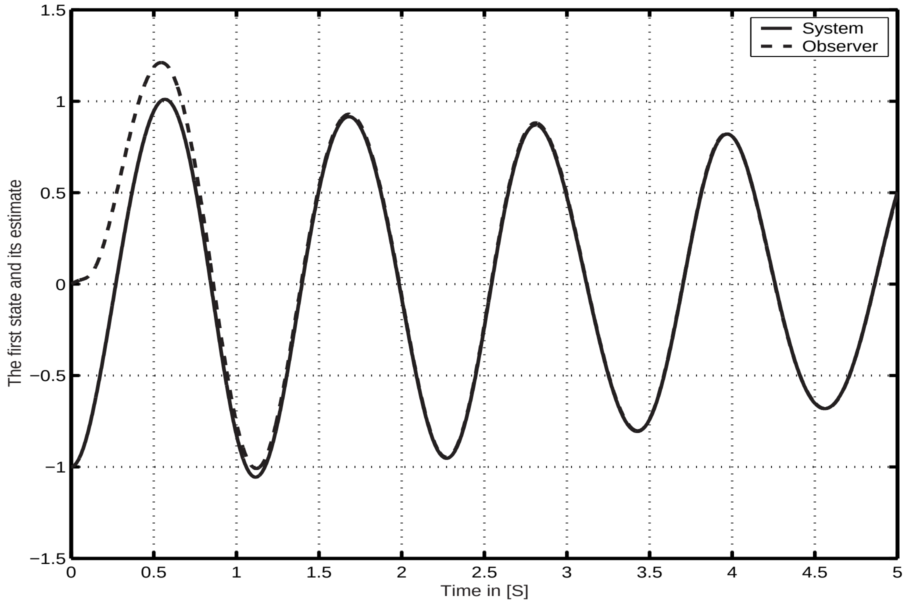

Figure 2: The first state and its estimate

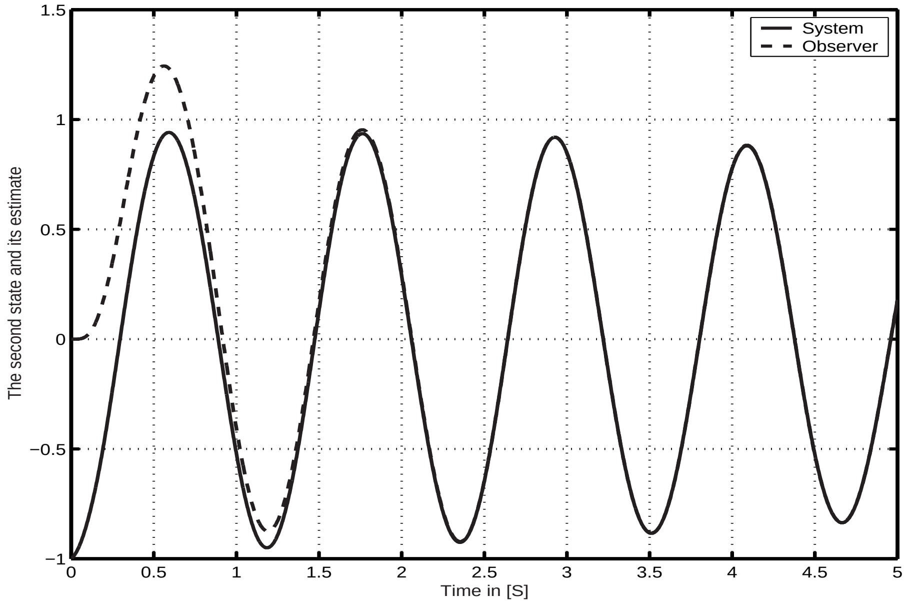

Figure 3: The second state and its estimate

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